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LETTER TO THE EDITOR

The effect of wavefront curvature on the coherence properties of laser light scattered by target centres in uniform motion

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Abstract. Moving speckle phenomena produced in the presence of wavefront curvature when laser light is scattered by an assembly of uniformly moving target centres are investigated. The first- and second-order space–time correlation functions of the scattered radiation are calculated and it is shown that the conditions for cross-spectral purity are not satisfied. It is pointed out that the effects offer a means of measurement of the velocity of the scatterers.

Perhaps the most familiar phenomenon associated with the scattering of laser light is the changing intensity or speckle pattern of bright and dark regions generated in the far field when such light is scattered by a moving rough surface. It is well known (for example, Rigden and Gordon 1962, Oliver 1963) that when a planar surface is illuminated normally by plane waves then the speckles merely 'evolve' or change their form as the surface is moved parallel to its plane, but that when the surface or the incident wavefront is curved then a translational motion of each speckle is observed before it loses its identity. This phenomenon has received a good deal of attention recently in connection with the assessment of the refraction performance of the eye (Ingelstam and Ragnarsson 1972 and references therein) and a number of elementary calculations have been performed to obtain an estimate of the speckle motion in terms of wavefront curvature and surface velocity (for example, Sporton 1969).

A similar effect, though invisible to the naked eye, will be produced by the interference of returns from different target centres when coherent light is scattered from uniformly moving seed particles in fluid flow, and could provide the basis for a laser anemometry system. In this letter, therefore, a quantitative mathematical description of the effects described above will be given in terms of the spatial and temporal coherence properties of the scattered light for the simple back-scattering geometry shown in figure 1. The primary objective will be to calculate the first- and second-order coherence

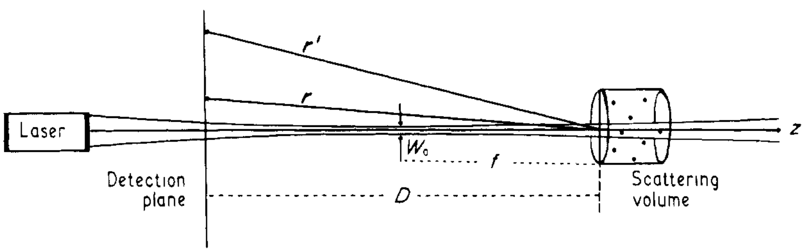

Figure 1. Back-scattering geometry.

functions

$$g^{(1)}(\boldsymbol{r}, \boldsymbol{r}'; \tau) = \langle \mathscr{E}^+(\boldsymbol{r}, t)\mathscr{E}^-(\boldsymbol{r}', t+\tau)\rangle / \surd(\langle I(\boldsymbol{r}, t)\rangle\langle I(\boldsymbol{r}', t+\tau)\rangle) \tag{1}$$

$$g^{(2)}(\boldsymbol{r}, \boldsymbol{r}'; \tau) = \langle I(\boldsymbol{r}, t)I(\boldsymbol{r}', t+\tau)\rangle / \langle I(\boldsymbol{r}, t)\rangle\langle I(\boldsymbol{r}', t+\tau)\rangle \tag{2}$$

where the intensity is given by

$$I(\boldsymbol{r}, t) = \mathscr{E}^+(\boldsymbol{r}, t)\mathscr{E}^-(\boldsymbol{r}, t) \tag{3}$$

in terms of $\mathscr{E}^+(\boldsymbol{r}, t)$, the positive frequency part of the field at a point $\boldsymbol{r}$ and time t. Stationarity will be assumed so that the angle brackets denote time or ensemble averages.

Provided that the distance, D, from the illuminated volume to the detection plane is much greater than the dimensions of the scattering volume, near the z axis $\mathscr{E}^+(\boldsymbol{r}, t)$ may be expressed as a sum over the totality, N, of scatterers in the form

$$\mathscr{E}^+(\boldsymbol{r}, t) = \sum_{j=1}^{N} \theta_j \alpha_j(\boldsymbol{r}, t)\mathscr{E}_0^+(\boldsymbol{R}_j, t) \exp[\mathrm{i}k(z_j + |\boldsymbol{r}_\perp - \boldsymbol{R}_{\perp j}|^2/2D)] \tag{4}$$

where a cylindrical polar coordinate system $R_j \equiv [\boldsymbol{R}_{\perp j}, z_j]$ etc has been adopted. In equation (4) $\mathscr{E}_0^+(\boldsymbol{R}_j, t)$ is the incident field (wavelength $\lambda = 2\pi/k$) at the jth scatterer situated at $\boldsymbol{R}_j(t)$, $\alpha_j(\boldsymbol{r}, t)$ is the corresponding scattering factor and θ_j takes the value one or zero according to whether $\boldsymbol{R}_j(t)$ lies within or outside the scattering volume Ω (Bourke *et al* 1970). Note that (4) is valid in the Fresnel region with respect to Ω since a curvature term has been retained in the exponent on the right-hand side (Born and Wolf 1965). Assuming that θ_j, R_j and α_j are statistically independent and that the scattering centres introduce path differences in excess of a wavelength so that the terms of (4) are statistically independent, the mean intensity may be written from (3) and (4) as follows:

$$\langle I(\boldsymbol{r}, t)\rangle = N_\Omega \langle|\alpha(\boldsymbol{r}, t)|^2\rangle\langle I_0(\boldsymbol{R}, t)\rangle_\Omega. \tag{5}$$

Here, N_Ω is the mean number of scatterers in the volume Ω, and the Ω-average is defined by

$$\langle F(\boldsymbol{R}, t)\rangle_\Omega = \frac{1}{\Omega}\int_\Omega \mathrm{d}^3 R F(\boldsymbol{R}, t)p(\boldsymbol{R}) \tag{6}$$

where $p(\boldsymbol{R})$ is the local number density and

$$N\langle\theta\rangle = \int_\Omega p(\boldsymbol{R})\,\mathrm{d}\boldsymbol{R} = N_\Omega. \tag{7}$$

A more explicit form may also be obtained for the coherence functions (1) and (2) using the relation

$$\boldsymbol{R}_j(t+\tau) = \boldsymbol{R}_j(t) + \mathbf{v}\tau \tag{8}$$

valid for scattering centres moving with uniform velocity $\mathbf{v}$. For stationary processes and $N \gg 1$ the following expressions are obtained:

$$g^{(1)}(\boldsymbol{r}, \boldsymbol{r}'; \tau) = \exp \mathrm{i}k[\tau \mathbf{v}_\perp . (\boldsymbol{U} + \boldsymbol{V}) - 2D\boldsymbol{U} . \boldsymbol{V} - v_\parallel \tau - \tfrac{1}{2}\mathbf{v}_\perp^2 \tau^2/D]$$
$$\times \frac{\langle \mathscr{E}_0^+(\boldsymbol{R}, t)\mathscr{E}_0^-(\boldsymbol{R} + \mathbf{v}\tau, t+\tau) \exp[\mathrm{i}k\boldsymbol{R}_\perp . (2\boldsymbol{V} - \mathbf{v}_\perp \tau/D)]\rangle_\Omega}{\langle I_0(\boldsymbol{R})\rangle_\Omega}$$
$$\times \frac{\langle\theta(t)\theta(t+\tau)\rangle}{\langle\theta\rangle} \frac{\langle\alpha(\boldsymbol{r}, t)\alpha^*(\boldsymbol{r}', t+\tau)\rangle}{(\langle|\alpha(\boldsymbol{r})|^2\rangle\langle|\alpha(\boldsymbol{r}')|^2\rangle)^{1/2}} \tag{9}$$

$$g^{(2)}(\boldsymbol{r},\boldsymbol{r}';\tau) = 1+|g^{(1)}(\boldsymbol{r},\boldsymbol{r}';\tau)|^2 + \frac{\langle I_0(\boldsymbol{R})I_0(\boldsymbol{R}+\mathbf{v}\tau)\rangle_\Omega}{\langle I_0(\boldsymbol{R})\rangle_\Omega^2}\,\frac{\langle\theta(t)\theta(t+\tau)\rangle}{N_\Omega\langle\theta\rangle}$$
$$\times\frac{\langle|\alpha(\boldsymbol{r},t)|^2|\alpha(\boldsymbol{r}',t+\tau)|^2\rangle}{\langle|\alpha(\boldsymbol{r})|^2\rangle\langle|\alpha(\boldsymbol{r}')|^2\rangle} \tag{10}$$

where

$$\boldsymbol{U} \simeq (\boldsymbol{r}_\perp+\boldsymbol{r}'_\perp)/2D, \qquad \boldsymbol{V} \simeq (\boldsymbol{r}'_\perp-\boldsymbol{r}_\perp)/2D, \tag{11}$$

$v_\parallel$ is the component of $\mathbf{v}$ parallel to the z axis and $\mathbf{v}_\perp$ the corresponding transverse component.

For large N_Ω (10) reduces to the usual factorization theorem associated with Gaussian light (Glauber 1963). On the other hand, when N_Ω is small the non-Gaussian or 'single-particle' contribution may be dominant. This is often the situation in laser anemometry experiments and the final term in (10) has been discussed at length in the literature (see for instance Abbiss *et al* 1974). The main concern of the present work, however, is the effect of wavefront curvature which enters (10) principally through the Ω-average appearing on the right-hand side of (9). Attention will therefore be concentrated, for the remainder of the letter, on the Gaussian contribution to (10) and for simplicity the factors in θ and α present in equation (9) will be set equal to unity. This will be the situation if the number and cross section of scattering centres do not fluctuate during delay times of interest.

In order to proceed further, a model for the scattering system and incident field must be adopted. Figure 1 shows a back-scattering configuration in which a laser beam propagating along the z axis is incident on a co-axial cylindrical scattering volume of radius ρ. The laser beam width W and radius of curvature σ are assumed to be constant within Ω so that $\mathscr{E}_0^+(\boldsymbol{R},t)$ can be represented by the function

$$\mathscr{E}_0^+(\boldsymbol{R},t) = E_0\exp -\mathrm{i}(\omega_0 t-kz-kR_\perp^2/2\sigma)\exp(-R_\perp^2/W^2). \tag{12}$$

In many situations of interest the scattering centres are uniformly distributed over the illuminated region. It is not difficult to evaluate (9) using (12) and the definition (6) in this case if $\rho \gg W$. The result obtained is

$$g^{(1)}(\boldsymbol{r},\boldsymbol{r}',\tau) = \exp[\mathrm{i}(2kD\boldsymbol{U}\,.\,\boldsymbol{V}+\omega_0\tau-2kv_\parallel\tau+k\mathbf{v}_\perp\,.\,\boldsymbol{U}\tau)]$$
$$\times\exp[-\tfrac{1}{2}(\mathbf{v}_\perp^2\tau^2/W^2+k^2W^2|\boldsymbol{V}-\kappa\mathbf{v}_\perp\tau|^2)] \tag{13}$$

where

$$\kappa = (D^{-1}+\sigma^{-1})/2. \tag{14}$$

The first-order correlation function of light back-scattered from a normally illuminated rough surface is also given by (13) which reduces to the expression obtained by Estes *et al* (1971) for this situation when $D \gg \sigma$ and $\boldsymbol{r}\equiv\boldsymbol{r}'$. Interpretation of the result in terms of moving speckle effects is more easily accomplished by writing the Gaussian part of the second-order, or intensity correlation function (10) in the form (figure 2)

$$g^{(2)}(\boldsymbol{r},\boldsymbol{r}';\tau) = 1+\exp[-(\tau-\tau_\mathrm{d})^2/\tau_\mathrm{c}^2]\exp[(\tau_\mathrm{d}/\tau_\mathrm{c})^2-k^2V^2W^2] \tag{15}$$

which for fixed $\boldsymbol{V}$ is a Gaussian function of τ of width

$$\tau_\mathrm{c} = W/v_\perp(1+k^2\kappa^2W^4)^{1/2} \tag{16}$$

centred at the delay time

$$\tau_\mathrm{d} = k^2\kappa W^4\mathbf{v}_\perp\,.\,\boldsymbol{V}/v_\perp^2(1+k^2\kappa^2W^4). \tag{17}$$

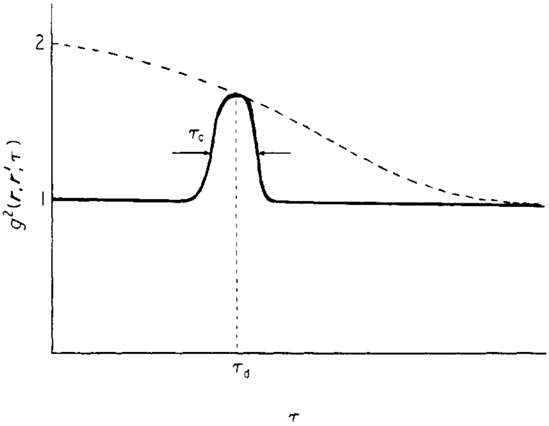

Figure 2. Typical second-order coherence function (Gaussian limit) in the presence of wavefront curvature. The envelope of the maximum as a function of detector separation (V) is shown by the broken curve.

At $\tau = 0$ the last term of (15) reduces to $\exp(-k^2V^2W^2)$ corresponding to a speckle size of the order of $2D/kW$. When $V = 0$ on the other hand it becomes $\exp(-\tau^2/\tau_c^2)$ indicating an intensity fluctuation time τ_c. In the absence of curvature (for example when the scattering volume is situated at the waist of the laser beam and detection is carried out in the Fraunhofer region) τ_c is determined by the amplitude fluctuations which arise as scattering centres move through the Gaussian intensity profile of the illuminating beam. This gives a transit time effect characterized by $\tau_c \sim W/v_\perp$ ($\kappa = 0$ in equation (16)) during which the speckles simply 'evolve'. When curvature effects are important, however, bodily motion of the intensity pattern produces additional fluctuations on a time scale determined by the motion and size of the speckles. A high degree of correlation then exists at a delay time τ_d equal to the time taken for an element of the pattern to traverse the distance separating the detection points r, r'. However, the effect will be degraded if τ_d exceeds $W/v_\perp$ because of the intrinsic changes or evolution of the speckle mentioned above. If the scattering volume lies sufficiently far from the laser beam focus or close to the detection plane $k\kappa W^2 \gg 1$ and curvature effects dominate. In this case, assuming for simplicity that V is parallel to $\mathbf{v}_\perp$ and writing $|V| = d/2D$ where d is the linear separation of the detection points, (17) reduces to

$$\tau_d = d/v_\perp(1 + D/\sigma) \tag{18}$$

corresponding to a speckle velocity $v_\perp(1 + D/\sigma)$. This result can, in fact, be deduced directly from simple geometrical considerations, and the limiting form obtained when $D \gg \sigma$ has appeared in previous publications (for example Sporton 1969). Note however, that according to (18) speckle motion should be observed in the Fresnel region even with plane-wave illumination ($D \ll \sigma$). Since the speckle size is $2D/kW$, the expected fluctuation time associated with bodily motion of the intensity pattern is $2D/kWv_\perp(1+D/\sigma)$ in agreement with equation (16). Moreover, the total distance travelled by a speckle during its lifetime $W/v_\perp$ is, from (18), $W(1+D/\sigma)$ and should be an indication of the value of d at which degradation of the correlation becomes significant. The last factor appearing in equation (15) reduces to $\exp[-d^2/W^2(1+D/\sigma)^2]$ when $k\kappa W^2 \gg 1$ and so confirms that this is indeed the case.

Having established that equations (15)–(17) are consistent with the observed moving speckle phenomenon, it is interesting to consider briefly some of the less familiar implications of the results (13) and (15). First it should be noted that, since the space and time variables in (13) are not separable, the conditions for cross-spectral purity (Mandel and Wolf 1965) are not satisfied. In this somewhat unusual situation spatial integration by a single detector of finite area will change the temporal characteristics of the detected signal. Conversely, if the scattered intensity is sampled with a finite integration time then the spatial coherence properties will be changed. It is not difficult to show that in addition to the expected reduction in noise, spatial integration will lead to a slowing down of the observed intensity fluctuations, whilst temporal integration will lead to an increase in the coherence length or effective speckle size of the intensity pattern in the direction of its motion.

Perhaps more important from a practical point of view is the detailed shape of the correlation function illustrated in figure 2. This suggests that by cross-correlating the output of two or more detectors the transverse velocity component $\mathbf{v}_\perp$ could be determined through a measurement of τ_d. The potential of such a technique can be more easily assessed by substituting the well known formulae for the width and curvature of a propagating laser beam (Kogelnik 1965) into (16) and (17). This gives

$$\tau_\mathrm{c} = W_0/v_\perp[(1+f/D)^2+(\pi W_0^2/\lambda D)^2]^{1/2} \tag{19}$$

$$\tau_\mathrm{d} = \frac{2\mathbf{v}_\perp \cdot \mathbf{V}f}{v_\perp^2}\left(\frac{1+f/D+(D/f)(\pi W_0^2/\lambda D)^2}{(1+f/D)^2+(\pi W_0^2/\lambda D)^2}\right) \tag{20}$$

where f is the distance from the scattering region to the laser beam waist of size W_0. A strong indication of the likely performance of a velocimeter based on measurement of τ_d will be the resolution $\tau_\mathrm{c}/\tau_\mathrm{d}$ and the peak-to-background ratio $g^{(2)}(\mathbf{r}, \mathbf{r}'; \tau_\mathrm{d})-1$. A particularly useful situation arises if the laser beam focus lies in the detection plane, $f = D$. Provided that $\pi W_0^2/\lambda D \gg 1$, τ_c and τ_d then become independent of distance from the scattering volume and

$$\tau_\mathrm{d}/\tau_\mathrm{c} = d/W_0; \qquad g^{(2)}(\mathbf{r}, \mathbf{r}'; \tau_\mathrm{d})-1 = \exp(-k^2 W_0^2 d^2/16D^2) \tag{21}$$

when $\mathbf{v}_\perp$ is parallel to $\mathbf{V}$ and d is the linear separation of the detection points as before. Thus, for example, if $d \sim 1$ cm, $W_0 \sim 10^{-2}$ cm and $D \sim 10^2$ cm ($W \sim 2$ mm) a resolution of 1:100 can be expected with a reduction of the correlation function peak height of less than 0·01! The possibility of achieving such a high resolution suggests that a laser anemometry technique based on the moving speckle phenomenon could prove useful in situations where a spread of velocities is present. This would give rise to an effect akin to spectral broadening.

Finally it is perhaps worth pointing out that multiple detector experiments can be avoided by mixing radiation using beam-splitter arrangements from different 'detection' points on the photocathode of a single detector, or by using a double beam or 'fringes' technique (Abbiss *et al* 1974). Both of these methods lead to contributions to the second-order coherence function of the detected signal proportional to the first-order correlation function (13).

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